Appendix of the paper 'Pricing, Allocation and Overbooking in Dynamic Service Network Competition when Demand is Uncertain'

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Abstract

We study the problem of combined pricing, resource allocation, and overbooking by service providers involved in dynamic non-cooperative oligopolistic competition on a network that represents the relationships of the providers to one another and to their customers when service demand is uncertain. We propose, analyze and compute solutions for a model that is more general than other models reported in the revenue management (RM) literature to date. In particular previous models typically consider only three or four of five key RM features that we have purposely built into our model: (1) pricing, (2) resource allocation, (3) dynamic competition, (4) an explicit network, and (5) uncertain demand. Illustrative realizations of the abstract problem we study are those of airline revenue management and service provision by companies facing resource constraints. Under fairly general regularity conditions, we prove existence and uniqueness of a pure-strategy Nash equilibrium for dynamic oligopolistic service network competition described by our model. We also show, again for an appropriate notion of regularity, that competition leads to the under-pricing of network services. We are able to numerically quantify the under-pricing gap for an illustrative example problem of intermediate size. Our proposed algorithm is shown to be implementable using well-known off-the-shelf commercial software.

Keywords : *dynamic games, variational inequalities, revenue management, pricing and allocation, overbooking*

APPENDIX : Proofs

Proof of Lemma 1. Note that the expected refunds and overbooking costs are separable in resource type. Taking the partial derivative of OBC_f with respect to $x^f_{j,N}$ we obtain

$$\begin{aligned}\frac{\partial OBC_f}{\partial x^f_{j,N}} &= \frac{\partial}{\partial z}\left\{\eta^f_j \cdot \sigma_Z\left[\phi(z) - z\left(1-\Phi(z)\right)\right]\right\}\cdot \frac{\partial z}{\partial x^f_{j,N}} + R^f_j \cdot \left(1-\alpha^f_j\right)\\ &= -\eta^f_j \cdot \sigma_Z\left(1-\Phi(z)\right)\cdot\frac{\partial z}{\partial x^f_{j,N}} + \frac{\eta^f_j}{2}\sqrt{\frac{\alpha^f_j\left(1-\alpha^f_j\right)}{x^f_{j,N}}}\left[\phi(z) - z\left(1-\Phi(z)\right)\right]\\ &\quad + R^f_j \cdot \left(1-\alpha^f_j\right)\end{aligned}$$

After substituting for σ_Z using

$$\sigma_Z = \sqrt{\alpha^f_j \cdot \left(1-\alpha^f_j\right)\cdot x^f_{j,N}}$$

and using the fact that z is normally distributed, one obtains after some simplification the following:

$$\begin{aligned}\frac{\partial OBC_f}{\partial x^f_{j,N}} &= \eta^f_j \alpha^f_j \cdot \left(1-\Phi(z)\right) + \frac{\eta^f_j \phi(z)}{2}\sqrt{\frac{\alpha^f_j\left(1-\alpha^f_j\right)}{x^f_{j,N}}} + R^f_j \cdot \left(1-\alpha^f_j\right) \qquad (1)\\ &> 0\end{aligned}$$

as required. ■

Proof of Proposition 1. Since $D^f_t(p_t) = d^f_t(p_t)\cdot z$;

$$\begin{aligned}E\left[p^f_t \cdot \min\left(u^f_{i,t}, D^f_t(p_t)\right)\right] &= \sum_{i\in\mathcal{S}} p^f_{i,t}\left[u^f_{i,t}\cdot \Pr(D^f_t \geq u^f_{i,t}) + d^f_t \int_0^{\frac{u^f_{i,t}}{d^f_{i,t}}} zf(z)dz\right]\\ &= \sum_{i\in\mathcal{S}} p^f_{i,t}\left[u^f_{i,t}\cdot \int_{\frac{u^f_{i,t}}{d^f_{i,t}}}^{\infty} zf(z)dz + d^f_{i,t}\int_0^{\frac{u^f_{i,t}}{d^f_{i,t}}} zf(z)dz\right]\\ &= \sum_{i\in\mathcal{S}} p^f_{i,t}\left[u^f_{i,t}\cdot\left(1-\int_0^{\frac{u^f_{i,t}}{d^f_{i,t}}} zf(z)dz\right) + d^f_{i,t}\int_0^{\frac{u^f_{i,t}}{d^f_{i,t}}} zf(z)dz\right]\\ &= \sum_{i\in\mathcal{S}} p^f_{i,t}u^f_{i,t} - \sum_{i\in\mathcal{S}} p^f_{i,t}d^f_{i,t}\int_0^{\frac{u^f_{i,t}}{d^f_{i,t}}}\left(\frac{u^f_{i,t}}{d^f_{i,t}} - z\right)f(z)dz\end{aligned}$$

using the identity $dF(z) = f(z)dz$

$$
\begin{aligned}
E\left[p_t^f \cdot \min\left(u_{i,t}^f, D_t^f\left(p_t\right)\right)\right] &= \sum_{i\in\mathcal{S}} p_{i,t}^f u_{i,t}^f - \sum_{i\in\mathcal{S}} p_{i,t}^f d_{i,t}^f \int_0^{\frac{u_{i,t}^f}{d_{i,t}^f}} \left(\frac{u_{i,t}^f}{d_{i,t}^f} - z\right) dF(z) \\
&= \sum_{i\in\mathcal{S}} \left\{ p_{i,t}^f u_{i,t}^f - p_{i,t}^f d_{i,t}^f \left[\frac{u_{i,t}^f}{d_{i,t}^f} \cdot F\left(\frac{u_{i,t}^f}{d_{i,t}^f}\right) - \int_0^{\frac{u_{i,t}^f}{d_{i,t}^f}} z dF(z)\right]\right\}
\end{aligned}
$$

integrating by parts the third term of the right hand side we obtain

$$
\begin{aligned}
E\left[p_t^f \cdot \min\left(u_{i,t}^f, D_t^f\left(p_t\right)\right)\right] &= \sum_{i\in\mathcal{S}} \left\{ \begin{array}{l} p_{i,t}^f u_{i,t}^f - \\ p_{i,t}^f d_{i,t}^f \left[\frac{u_{i,t}^f}{d_{i,t}^f} \cdot F\left(\frac{u_{i,t}^f}{d_{i,t}^f}\right) - zF(z)\Big|_0^{\frac{u_{i,t}^f}{d_{i,t}^f}} + \int_0^{\frac{u_{i,t}^f}{d_{i,t}^f}} F(z)dz\right] \end{array} \right\} \\
&= \sum_{i\in\mathcal{S}} p_{i,t}^f u_{i,t}^f - p_{i,t}^f d_{i,t}^f \int_0^{\frac{u_{i,t}^f}{d_{i,t}^f}} F(z)dz
\end{aligned}
$$

Hence the proof. ■

Proof of Lemma 2. To show $H_{f,t}$ is strictly concave in u_t^f, we need to establish

$$
H_{f,t}\left(p_t^f, u^\mu; \lambda^f; p_t^{-f}; t\right) > \mu H_f\left(p_t^f, u_1; \lambda^f; p_t^{-f}; t\right) + (1-\mu) H_f\left(p_t^f, u_2; \lambda^f; p_t^{-f}; t\right)
$$

where $u^\mu = \mu u_1 + (1-\mu) u_2$ with $\mu \in [0,1]$ with is same as

$$
\int_0^{\frac{u^\mu}{d_{i,t}^f(p_{i,t})}} F(\tau)\, d\tau < \mu \int_0^{\frac{u_1}{d_{i,t}^f(p_{i,t})}} F(\tau)\, d\tau + (1-\mu) \int_0^{\frac{u_2}{d_{i,t}^f(p_{i,t})}} F(\tau)\, d\tau
$$

Therefore it will suffice if we can show that $\int_0^{\frac{u_{i,t}^f}{d_{i,t}^f(p_{i,t})}} F(\tau)\, d\tau$ is strictly convex in $u_{i,t}^f$ which is true as

$$
\frac{\partial}{\partial u_{i,t}^f} \int_0^{\frac{u_{i,t}^f}{d_{i,t}^f(p_{i,t})}} F(\tau)\, d\tau = \frac{1}{d_{i,t}^f(p_{i,t})} F\left(\frac{u_{i,t}^f}{d_{i,t}^f(p_{i,t})}\right) > 0
$$

which completes the proof. ■

Proof of Lemma 3. Part (a) : In Lemma 1 we have already established

$$
\frac{\partial OBC_j}{\partial x_{j,N}^f} > 0
$$

If it can be shown that $\partial x_{j,N}^f \Big/ \partial \lambda_i^f > 0$ for all $i \in \mathcal{C}$, this will imply

$$
\frac{\partial OBC_j}{\partial \lambda_k^f} = \frac{\partial OBC_j}{\partial x_{j,N}^f} \cdot \frac{\partial x_{j,N}^f}{\partial \lambda_l^f} > 0 \text{ for all } l \in \mathcal{C}
$$

Differentiating $x_{j,N}^f$ w.r.t. λ_l^f using the expression (same as the eqn (4) of the main paper)

$$x_{j,N}^f = \sum_{k=0}^{N-1} \sum_{i \in \mathcal{S}} \left\{ a_{ij} \cdot d_{i,k}^f \left(p_{i,k}\right) \cdot F^{-1} \left(\frac{p_{i,k}^f + \mathcal{A}_i^T \cdot \lambda^f}{p_{i,k}^f} \right) \right\} \text{ for all } j \in \mathcal{C} \tag{2}$$

we have

$$\begin{aligned} \frac{\partial x_{j,N}^f}{\partial \lambda_l^f} &= \sum_{k=0}^{N-1} \sum_{i \in \mathcal{S}} \left\{ a_{ij} \frac{d_{i,k}^f \left(p_{i,k}\right)}{f\left(F^{-1}\left(\frac{p_{i,k}^f + \mathcal{A}_i^T \cdot \lambda^f}{p_{i,k}^f}\right)\right)} \cdot \frac{1}{p_{i,k}^f} \right\} \\ &> 0 \end{aligned}$$

The last inequality is obtained as the pdf $f(\cdot) \geq 0$, a_{ij} is either 0 or 1. This concludes the first part of the proof.

Part (b) : Differentiating $x_{j,N}^f$ w.r.t. $p_{i,t}^f$ using (2)

$$\frac{\partial x_{j,N}^f}{\partial p_{i,t}^f} = a_{ij} \left[\frac{\partial d_{i,t}^f}{\partial p_{i,t}^f} F^{-1} \left(1 + \frac{\mathcal{A}_i^T \cdot \lambda^f}{p_{i,t}^f} \right) - \frac{d_{i,t}^f}{f\left(F^{-1}\left(1 + \frac{\mathcal{A}_i^T \cdot \lambda^f}{p_{i,t}^f}\right)\right)} \frac{\mathcal{A}_i^T \cdot \lambda^f}{\left(p_{i,t}^f\right)^2} \right] \tag{3}$$

where $a_{ij} = 1$ if service i utilizes resource j and 0 otherwise. Let $F^{-1}\left(1 + \frac{\mathcal{A}_i^T \cdot \lambda^f}{p_{i,t}^f}\right) = y$ which implies $F(y) = 1 + \frac{\mathcal{A}_i^T \cdot \lambda^f}{p_{i,t}^f}$, and

$$\frac{dy}{dp_{i,t}^f} = -\frac{1}{f(y)} \frac{\mathcal{A}_i^T \cdot \lambda^f}{\left(p_{i,t}^f\right)^2} \geq 0$$

as $\lambda^f \leq 0$. Rewriting (3), we obtain

$$\frac{\partial x_{j,N}^f}{\partial p_{i,t}^f} = a_{ij} \frac{d_{i,t}^f}{p_{i,t}^f} \left[-e_{i,t}^f \cdot y + \frac{\left[1 - F\left(y\right)\right]}{f(y)} \right] \tag{4}$$

Using the definition of generalized failure rate $\beta(y) = \frac{y f(y)}{1 - F(y)}$, (4) can be further simplified as

$$\frac{\partial x_{j,N}^f}{\partial p_{i,t}^f} = a_{ij} \frac{d_{i,t}^f}{p_{i,t}^f} y \left[-e_{i,t}^f + \frac{1}{\beta(y)} \right]$$

Therefore,

$$\begin{aligned} \frac{\partial OBC_j}{\partial p_{i,t}^f} &= \sum_{j \in \mathcal{C}} \frac{\partial OBC_j}{\partial x_{j,N}^f} \cdot \frac{\partial x_{j,N}^f}{\partial p_{i,t}^f} \\ &= \sum_{j \in \mathcal{C}} a_{ij} \frac{\partial OBC_j}{\partial x_{j,N}^f} \cdot \frac{d_{i,t}^f}{p_{i,t}^f} y \left[-e_{i,t}^f + \frac{1}{\beta(y)} \right] \end{aligned}$$

From Lemma 1 we have $\partial OBC_j \Big/ \partial x^f_{j,N} > 0$ and from item 5 of Assumption A2, $e^f_{i,t}$ is increasing in $p^f_{i,t}$. Finally, from Assumption A1 $\frac{d\beta(y)}{dy} > 0$ thus

$$\frac{d}{dp^f_{i,t}}\left(\frac{1}{\beta(y)}\right) = -\frac{1}{\beta^2(y)} \cdot \frac{d\beta(y)}{dy} \cdot \frac{dy}{dp^f_{i,t}} < 0 \tag{5}$$

Hence the proof. ■

Proof of Lemma 4. We observe that the Hamiltonian is separable, i.e., $H_{f,t} = \sum_{i\in\mathcal{S}} H^i_{f,t}$ where

$$H^i_{f,t} = p^f_{i,t} \cdot d^f_{i,t}(p_{i,t}) \int_0^{F^{-1}\left(\mu^f_{i,t}\right)} \tau f(\tau)\, d\tau$$

and $H^i_{f,t}$ depends only on $p^f_{i,t}$ for given non-own prices p^{f-}_t and own shadow prices λ^f of resources. Thus

$$\frac{\partial H_{f,t}}{\partial p^f_{i,t}} = \frac{\partial H^i_{f,t}}{\partial p^f_{i,t}}$$

Taking partials of $H^i_{f,t}$ w.r.t. $p^f_{i,t}$ for some $i \in \mathcal{S}$ we obtain

$$\begin{aligned}
\frac{\partial H^i_{f,t}}{\partial p^f_{i,t}} &= \left(p^f_{i,t}\frac{\partial d^f_{i,t}}{\partial p^f_{i,t}} + d^f_{i,t}\right) \cdot \int_0^{F^{-1}\left(\mu^f_{i,t}\right)} \tau f(\tau)\, d\tau - d^f_{i,t}\frac{A_i^T \cdot \lambda^f}{p^f_i} \cdot F^{-1}\left(\mu^f_{i,t}\right) \\
&= d^f_{i,t} \cdot \int_0^{F^{-1}\left(\mu^f_{i,t}\right)} \tau f(\tau)\, d\tau \left[\left(-e^f_{i,t} + 1\right) - \frac{A_i^T \cdot \lambda^f}{p^f_i} \cdot \frac{F^{-1}\left(\mu^f_{i,t}\right)}{\int_0^{F^{-1}\left(\mu^f_{i,t}\right)} \tau f(\tau)\, d\tau}\right] \\
&= \frac{H^i_{f,t}}{p^f_{i,t}} \left[\left(-e^f_{i,t} + 1\right) - \frac{A_i^T \cdot \lambda^f}{p^f_i} \cdot \frac{F^{-1}\left(\mu^f_{i,t}\right)}{\int_0^{F^{-1}\left(\mu^f_{i,t}\right)} \tau f(\tau)\, d\tau}\right] \qquad (6)
\end{aligned}$$

where $e^f_{i,t}$ is the local price elasticity. Next, we observe

$$\begin{aligned}
\frac{\partial^2 H_{f,t}}{\partial p^f_{i,t}\partial p^f_{j,t}} &= \frac{\partial^2 H^i_{f,t}}{\partial p^f_{i,t}\partial p^f_{j,t}} = 0 \text{ for all } j \neq i \qquad (7) \\
\frac{\partial^2 H_{f,t}}{\partial p^f_{i,t}\partial p^g_{j,t}} &= \frac{\partial^2 H^i_{f,t}}{\partial p^f_{i,t}\partial p^g_{j,t}} = 0 \text{ for all } j \neq i, g \neq f \qquad (8)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 H_{f,t}}{\partial p^f_{i,t}\partial p^g_{i,t}} &= -\frac{H^i_{f,t}}{p^f_{i,t}}\frac{\partial e^f_{i,t}}{\partial p^g_{i,t}} \\
&\geq 0 \text{ (from item 7 of assumption A2)}
\end{aligned}$$

Hence the proof. ■

Proof of Lemma 5. We have seen in Lemma 4 that for a given λ^f (thus c^f) the game is supermodular. In addition, if we are able to show that $H_{f,t}$ has increasing differences in $\left(p_t^f, c^f\right)$ for each p_t^{-f}, we can use Theorem 6 to establish that the extremal equilibria of the revenue optimization game are increasing functions of rhe shadow price of resources, λ^f. Differentiating $H_{f,t}$ by c_j^f we obtain

$$\frac{\partial H_{f,t}}{\partial c_i^f} = -d_{i,t}^f \cdot F^{-1}\left(1 - \frac{c_i^f}{p_{i,t}^f}\right) \tag{9}$$

Differentiating again (9) w.r.t. service prices we observe $\partial^2 H_{f,t} \Big/ \partial c_i^f \partial p_{j,t}^f = 0$ if $j \neq i$ and

$$\begin{aligned}
\frac{\partial^2 H_{f,t}}{\partial c_i^f \partial p_{i,t}^f} &= -\frac{\partial d_{i,t}^f}{\partial p_{i,t}^f} \cdot F^{-1}\left(1 - \frac{c_i^f}{p_{i,t}^f}\right) - \frac{d_{i,t}^f}{f\left(F^{-1}\left(1 - \frac{c_i^f}{p_{i,t}^f}\right)\right)} \cdot \frac{c_i^f}{\left(p_{i,t}^f\right)^2} \\
&= \frac{d_{i,t}^f}{p_{i,t}^f}\left[e_{i,t}^f \cdot F^{-1}\left(1 - \frac{c_i^f}{p_{i,t}^f}\right) - \frac{1}{f\left(F^{-1}\left(1 - \frac{c_i^f}{p_{i,t}^f}\right)\right)} \cdot \frac{c_i^f}{p_{i,t}^f}\right]
\end{aligned} \tag{10}$$

Let $F^{-1}\left(1 - \frac{c_i^f}{p_{i,t}^f}\right) = y$ which implies $\frac{c_i^f}{p_{i,t}^f} = 1 - F(y)$, and

$$\frac{dy}{dp_{i,t}^f} = +\frac{1}{f(y)} \frac{c_i^f}{\left(p_{i,t}^f\right)^2} \geq 0$$

as $c_i^f = -\left(\lambda^f\right)^T \mathcal{A}_i \geq 0$. Rewriting (10) we obtain

$$\begin{aligned}
\frac{\partial^2 H_{f,t}}{\partial c_i^f \partial p_{i,t}^f} &= \frac{d_{i,t}^f}{p_{i,t}^f}\left[e_{i,t}^f \cdot y - \frac{1 - F(y)}{f\left(y\right)}\right] \\
&= \frac{d_{i,t}^f}{p_{k,t}^f} y \left[e_{i,t}^f - \frac{1}{\beta\left(y\right)}\right]
\end{aligned}$$

Now, we know $y \geq 0$, $e_{i,t}^f$ is increasing in $p_{i,t}^f$ (item 5 of assumption A2) and $\frac{1}{\beta(y)}$ is strictly decreasing in $p_{i,t}^f$ from (5). Further, since from the condition $e_{i,t}^f\Big|_{p_{i,t}^f = p_{,\min}^f} \geq \frac{1}{\beta\left(F^{-1}\left(1 + \frac{\mathcal{A}_i^T \cdot \lambda^f}{p_{i,\min}^f}\right)\right)}$ we know $\left.\frac{\partial^2 H_{f,t}}{\partial c_i^f \partial p_{i,t}^f}\right|_{p_{i,t}^f = p_{i,\min}^f} \geq 0$ and more over

$$\frac{\partial^2 H_{f,t}}{\partial c_i^f \partial p_{i,t}^f} \geq 0 \text{ for all } p_{i,\min}^f \leq p_{i,t}^f \leq p_{i,\max}^f \text{ and } c_i^f \geq 0$$

Therefore, condition (ii) of Theorem 1 (Theorem 7; Amir 2003) is satisfied as well, hence the extremal equilibria of the game are increasing functions of c^f.■

Proof of Lemma 6. Since the Hamiltonian is separable and from (6)

$$\frac{\partial H^i_{f,t}}{\partial p^f_{i,t}} = \frac{H^i_{f,t}}{p^f_{i,t}} \left[\left(-e^f_{i,t} + 1 \right) - \frac{A^T_i \cdot \lambda^f}{p^f_i} \cdot \frac{F^{-1}\left(\mu^f_{i,t}\right)}{\int_0^{F^{-1}\left(\mu^f_{i,t}\right)} \tau f\left(\tau\right) d\tau} \right]$$

where $e^f_{i,t}$ is the local price elasticity. Let us define

$$\psi\left(p^f_{i,t}, p^{f-}_t; \lambda^f\right) = -\frac{A^T_i \cdot \lambda^f}{p^f_i} \cdot \frac{F^{-1}\left(\mu^f_{i,t}\right)}{\int_0^{F^{-1}\left(\mu^f_{i,t}\right)} \tau f\left(\tau\right) d\tau}$$

Now $\frac{\partial H_{f,t}}{\partial p^f_{i,t}} = 0$ is equivalent to $e^f_{i,t} = 1 + \psi\left(p^f_{i,t}, p^{f-}_t; \lambda^f\right)$. We know $e^f_{i,t}$ is increasing in $p^f_{i,t}$ (from item 5 of assumption A2), therefore if, in addition, we can show that $\psi\left(p^f_{i,t}, p^{f-}_t; \lambda^f\right)$ is strictly decreasing in $p^f_{i,t}$, then $e^f_{i,t} = 1 + \psi\left(p^f_{i,t}, p^{f-}_t; \lambda^f\right)$ can have at most one solution. We will also then be able to show that for given p^{f-}_i and λ^f, there exists some p^{f*}_i such that $H_{f,t}$ is nondecreasing for $p^f_i < p^{f*}_i$ and nonincreasing for $p^f_i > p^{f*}_i$, hence quasi-concave. Let $y = F^{-1}\left(\mu^f_{i,t}\right)$, then $-\frac{A^T \cdot \lambda^f}{p^f_i} = 1 - F(y)$ and $\frac{dy}{dp^f_i} > 0$ for $A^T_i \cdot \lambda^f < 0$. Taking the derivative of $\psi\left(p^f_{i,t}, p^{f-}_t; \lambda^f\right)$ w.r.t. p^f_i

$$\frac{\partial \psi\left(p^f_{i,t}, p^{f-}_t; \lambda^f\right)}{\partial p^f_{i,t}} = \left[(1 - \beta(y)) \cdot \int_0^y \tau f\left(\tau\right) d\tau - y^2 f(y) \right] \frac{1 - F\left(y\right)}{\left[\int_0^y \tau f\left(\tau\right) d\tau\right]^2} \cdot \frac{dy}{dp^f_i} \tag{11}$$

where $\beta(y)$ is a generalized failure rate. Now consider

$$\delta\left(y\right) = (1 - \beta(y)) \cdot \int_0^y \tau f\left(\tau\right) d\tau - y^2 f(y)$$

where it is evident that $\delta\left(0\right) = 0$. If we can establish that $\delta'\left(y\right) < 0$ for all $y > 0$, then we will be able to establish that $\delta\left(y\right) < 0$ for all $y > 0$, which will also ensure that $\frac{\partial \psi\left(p^f_{i,t}, p^{f-}_t; \lambda^f\right)}{\partial p^f_{i,t}} < 0$ i.e., $\psi\left(\cdot\right)$ is decreasing in $p^f_{i,t}$. Re-arranging terms we arrive

$$\delta\left(y\right) = (1 - \beta(y)) \cdot \int_0^y \tau f\left(\tau\right) d\tau - y\beta(y) \cdot \left[1 - F\left(y\right)\right] \tag{12}$$

where we have used the identity $\beta(y)\left[1 - F\left(y\right)\right] = yf(y)$. Taking the derivative of $\delta\left(y\right)$ w.r.t. y

$$\delta'\left(y\right) = -\beta'(y) \cdot \int_0^y \tau f\left(\tau\right) d\tau - y\beta'(y) \cdot \left[1 - F\left(y\right)\right]$$

here once again we have utilized the identity. Since $\beta'(y) > 0$ from IGFR assumption, $\delta'(y) < 0$ for all $y > 0$ which completes the proof. ■

Proof of Theorem 2. It is relatively straightforward to show that a policy p^* that solves the variational inequality problem: find $p_{i,t}^{f*} \in \Lambda_f$ such that

$$\left[\nabla_{p_t^f} H_f \left(p_t^{f*}; \lambda^f; p_t^{-f}; t\right)\right]^T \cdot \left(p_t^f - p_t^{f*}\right) \leq 0 \tag{13}$$

(same as eqn (22) of the paper) for each firm $f \in \mathcal{F}$ simultaneously, also solves the joint variational inequality problem: find $p^* \in \mathcal{K}$ such that

$$\left(\begin{array}{c} \nabla_{p^1} H_1 \left(p^{1*}; \lambda^{1*}; p^{-1*}\right) \\ \vdots \\ \nabla_{p^{|\mathcal{F}|}} H_{|\mathcal{F}|} \left(p^{|\mathcal{F}|*}; \lambda^{|\mathcal{F}|*}; p^{-|\mathcal{F}|*}\right) \end{array}\right)^T \cdot \left(\begin{array}{c} p^1 - p^{1*} \\ \vdots \\ p^{|\mathcal{F}|} - p^{|\mathcal{F}|*} \end{array}\right) \leq 0 \tag{14}$$

(same as eqn (25) of the paper). We will now show the converse, i.e., the solution to joint variational inequality problem (14) solves variational inequality problems (13) for each firm f simultaneously. That is, if p^* is a solution to joint VI problem (14), then for each firm $f \in \mathcal{F}$, p^{f*} solves the variational inequality problem (13) with competitors' policies p^{-f} given by p^{-f*}. Own shadow price is computed by solving the equation

$$\lambda^{f*} = -\nabla_{x_N^f} OBC_f \left(p^{f*}, p^{-f*}, \lambda^{f*}\right)$$

Note that (14) is equivalent to the following fictitious mathematical program

$$\begin{aligned} \max G(p) &= \sum_{t=0}^{N-1} \sum_{f \in \mathcal{F}} \sum_{i \in \mathcal{S}} \frac{\partial H_f \left(p_t^{f*}; \lambda^{f*}; p_t^{-f*}\right)}{\partial p_{i,t}^f} p_{i,t}^f && (15) \\ \text{subject to} \quad p_{\min}^f &\leq p_t^f \leq p_{\max}^f \text{ for all } f \in \mathcal{F}, t \in [0, N-1] && (16) \\ \lambda^f &= -\nabla_{x_N^f} OBC_f \quad \text{for all } f \in \mathcal{F} && (17) \end{aligned}$$

where it is essential to recognize that $G(p)$ is a linear functional that assumes knowledge of the solution of (14); as such $G(p)$ is a mathematical construct for use in analysis and has no meaning as a computational device. The corresponding necessary and sufficient conditions for this mathematical program are identical to (13) for all $f \in \mathcal{F}$ as because

$$\frac{\partial G^*}{\partial p_{i,t}^f} = \frac{\partial H_f \left(p_t^{f*}; \lambda^{f*}; p_t^{-f*}\right)}{\partial p_{i,t}^f}$$

where

$$G^* = G(p^*) = \sum_{t=0}^{N-1} \sum_{f \in \mathcal{F}} \sum_{i \in \mathcal{S}} \frac{\partial H_f \left(p_t^{f*}; \lambda^{f*}; p_t^{-f*}\right)}{\partial p_{i,t}^f} p_{i,t}^{f*}$$

hence the proof. ■

Proof of Theorem 3. We need to establish that there exists at least one solution of the VI (14). Since any solution of (14) is a Nash equilibrium of the game (per Theorem 2), then that solution will also be a Nash equilibrium of the game. Note that the strategy space of each firms' pricing decision for each service is a closed interval, hence p is a non-empty, compact and convex set of $\mathbb{R}^{|\mathcal{F}|\times|\mathcal{S}|\times(N-1)}$. Further, $\begin{pmatrix} \nabla_{p^1} H_1 & \cdots & \nabla_{p^{|\mathcal{F}|}} H_{|\mathcal{F}|} \end{pmatrix}^T$ is a continuous mapping from $\mathcal{K}$ into $\mathbb{R}^{|\mathcal{F}|\times|\mathcal{S}|\times(N-1)}$. Therefore, invoking Theorem 3.1 of Harker and Pang (1990) we establish that there exists a solution of (14), hence the proof. ■

Proof of Theorem 4. To establish the claim, we should be able to establish

$$\frac{\partial \tilde{H}_t^c\left(p_t^*, \lambda^*, t\right)}{\partial p_{i,t}^f} \geq 0 \text{ for all } i \in \mathcal{S}, f \in \mathcal{F}, t \in [0, N-1] \tag{18}$$

and

$$\frac{\partial H_t^f\left(\tilde{p}_t^f, \tilde{\lambda}^f, \tilde{p}_t^{-f}, t\right)}{\partial p_{i,t}^f} \leq 0 \text{ for all } i \in \mathcal{S}, f \in \mathcal{F}, t \in [0, N-1] \tag{19}$$

To establish (18), we will consider 2 cases : (a) $p_{i,t}^{f*} = p_{i,\min}^f$ and (b) $p_{i,t}^{f*} > p_{i,\min}^f$.

Case (a) : When $p_{i,t}^{f*} = p_{i,\min}^f$, since cooperative equilibrium has also the same bounds on service prices, $\tilde{p}_{i,t}^f \geq p_{i,\min}^f (= p_{i,t}^{f*})$

Case (b) : On the other hand, when $p_{i,t}^{f*} > p_{i,\min}^f$ we need to establish (18). From (**??**)

$$\frac{\partial \tilde{H}_t^c\left(p_t^*, \lambda^*, t\right)}{\partial p_{i,t}^f} = \frac{\partial H_t^f\left(p_t^{f*}, \lambda^{f*}, p_t^{-f*}, t\right)}{\partial p_{i,t}^f} + \sum_{g \neq f} \frac{\partial H_t^g\left(p_t^{g*}, \lambda^{g*}, p_t^{-g*}, t\right)}{\partial p_{i,t}^f}$$

since $p_{i,t}^{f*} > p_{i,\min}^f$ therefore $\frac{\partial H_t^f\left(p_t^{f*}, \lambda^{f*}, p_t^{-f*}, t\right)}{\partial p_{i,t}^f} \geq 0$ (if $p_{i,t}^{f*}$ is a strictly interior point, the partial will be equal to 0 and strictly positive if $p_{i,t}^{f*} = p_{i,\max}^f$). Further,

$$\begin{aligned} \frac{\partial H_t^g\left(p_t^g, \lambda^g, p_t^{-g}, t\right)}{\partial p_{i,t}^f} &= p_{i,t}^g \cdot \frac{\partial d_{i,t}^g}{\partial p_{i,t}^f} \cdot \int_0^{F^{-1}\left(\mu_{i,t}^g\right)} \tau f(\tau)\, d\tau \\ &\geq 0 \text{ (from item 6 of assumption A2)} \end{aligned} \tag{20}$$

therefore

$$\frac{\partial \tilde{H}_t^c\left(p_t^*, \lambda^*, t\right)}{\partial p_{i,t}^f} \geq 0$$

To establish (19), we once again consider 2 cases : (c) $\tilde{p}_{i,t}^f = p_{i,\max}^f$ and (d) $\tilde{p}_{i,t}^f < p_{i,\max}^f$.

Case (c) : When $\tilde{p}_{i,t}^f = p_{i,\max}^f$, by definition the non-coobperative equilibrium cannot be greater than $\tilde{p}_{i,t}^f$.

Case (d) : We need to establish (19) when $\tilde{p}^f_{i,t} < p^f_{i,\max}$. At this cooperative equilibrium point

$$\frac{\partial \tilde{H}^c_t\left(\tilde{p}_t, \tilde{\lambda}, t\right)}{\partial p^f_{i,t}} \leq 0$$

In particular, if $\tilde{p}^f_{i,t}$ is a strictly interior point, the partial will be equal to 0 and strictly positive if $\tilde{p}^f_{i,t} = p^f_{i,\min}$. Using (**??**)

$$\frac{\partial H^f_t\left(\tilde{p}^f_t, \tilde{\lambda}^f, \tilde{p}^{-f}_t, t\right)}{\partial p^f_{i,t}} + \sum_{g \neq f} \frac{\partial H^g_t\left(\tilde{p}^g_t, \tilde{\lambda}^g, \tilde{p}^{-g}_t, t\right)}{\partial p^f_{i,t}} \leq 0$$

From (20) we know every term inside the summation are non-negative; thus

$$\frac{\partial H^f_t\left(\tilde{p}^f_t, \tilde{\lambda}^f, \tilde{p}^{-f}_t, t\right)}{\partial p^f_{i,t}} \leq 0$$

Therefore, (18) says that at the non-cooperative Nash equilibrium point, the joint profit can be further increased if all firms can collude. But no firm will take such strategy unilaterally because it has already made the best response given other firms' pricing decisions. Further, (19) says that if firms adopt cooperative strategies while they are actually involved in non-cooperative equilibrium, they have an incentive to decrease prices to attract more demand. Thus cooperative strategy is clearly not their best response strategy and is not a Nash equilibrium. ■

Proof of Theorem 5. The fixed point problem considered requires that

$$p = \arg\min_q \left\{ \frac{1}{2} \left\| p - \alpha \cdot F\left(p, \lambda, t\right) - q \right\|^2 : q \in \mathcal{K} \right\} \tag{21}$$

where $\alpha \in \Re^1_{++}$. That is, we seek the solution of the following mathematical program

$$\min_q J(q) = \frac{1}{2} \left[p - \alpha \cdot F\left(p, \lambda, t\right) - q \right]^2$$

subject to

$$q \in \mathcal{K}$$

Let us take $q^* \in \mathcal{K}$ be a minimum of the above finite dimensional mathematical program and recall that $\mathcal{K}$ is convex. Since $J(q)$ is convex and differentiable at $q^* \in \mathcal{K}$, a necessary and sufficient condition is

$$\left\langle \nabla J\left(q^*\right), q - q^* \right\rangle \geq 0 \text{ for all } q \in \mathcal{K} \tag{22}$$

further

$$\nabla J\left(q^*\right) = (-1) \left[p - \alpha \cdot F\left(p, \lambda, t\right) - q^* \right] \tag{23}$$

By virtue of (21) $p = q^*$, so (23) may be restated as

$$\nabla J(q^*) = \alpha \cdot F(q^*, \lambda^*, t)$$

where λ^* is obtained by solving the equation

$$\lambda_j^f = -\frac{\partial OBC_f}{\partial x_{j,N}^f} \text{ for all } j \in \mathcal{C}$$

(which is same as eqn (16) of the paper) for given q^*. Taken together, (22) and (23) give

$$\alpha \langle F(q^*, \lambda^*, t), q - q^* \rangle \geq 0 \text{ for all } q \in \mathcal{K} \tag{24}$$

Because α is positive and constant and $F(p^*, \lambda^*, t) = \left(\ -\nabla_{p^f} H_f \left(p^{f*}; \lambda^*; p^{-f*}\right) \ \right)_{f \in \mathcal{F}}$, (24) reduces to

$$\sum_{f \in \mathcal{F}} \left[\nabla_{q^f} H_f \left(q^{f*}; \lambda^*; q^{-f*}\right)\right]^T \cdot \left(q^f - q^{f*}\right) \leq 0 \text{ for all } q \in \mathcal{K}$$

(14) follows immediately, and the theorem is proved. ■

Proof of Theorem 6. We need to study the negative of the Jacobian matrix. If we can establish that at the point $p^{\kappa*}$ where

$$\left(\nabla_{p^{f\kappa}} H_f \left(p^{f,\kappa*}; \lambda^\kappa; p^{-f,\kappa*}\right)\right)_{f \in \mathcal{F}} = 0 \tag{25}$$

(the above eqn is same as eqn (44) of the paper) holds the diagonal terms of the negative Jacobian matrix are strictly positive, off-diagonal terms are nonnegative and the matrix is strictly diagonally dominant, it will follow automatically that the negative Jacobian matrix has all principal minors positive. Hence (25) has an unique solution. We know

$$\begin{aligned}
-\frac{\partial^2 H_{f,t}}{\partial p_{i,t}^f \partial p_{j,t}^f} &= 0 \text{ for all } i \neq j, f \in \mathcal{F} \text{ and } t \in [0, N-1] \\
-\frac{\partial^2 H_{f,t}}{\partial p_{i,t}^f \partial p_{j,t}^g} &= 0 \text{ for all } i \neq j, f \neq g \text{ and } t \in [0, N-1] \\
-\frac{\partial^2 H_{f,t}}{\partial p_{i,t}^f \partial p_{i,t}^g} &= \frac{H_{f,t}}{p_{i,f}^f} \cdot \frac{\partial e_{i,t}^f}{\partial p_{i,t}^g} - \frac{1}{p_{i,t}^f} \frac{\partial d_{i,t}^f}{\partial p_{i,t}^g} \left[\left(-e_{i,t}^f + 1\right) - \frac{A_i^T \cdot \lambda^f}{p_i^f} \cdot \frac{F^{-1}\left(\mu_{i,t}^f\right)}{\int_0^{F^{-1}\left(\mu_{i,t}^f\right)} \tau f(\tau)\, d\tau}\right] \qquad (26) \\
&\leq 0 \text{ for all } i \in \mathcal{S}, f \neq g \text{ and } t \in [0, N-1]
\end{aligned}$$

The last inequality is obtained because at $p^{\kappa*}$, $\frac{\partial H_{f,t}}{\partial p_{i,t}^f} = 0$ (therefore terms inside the bracket of (26) vanish). Further, by Lemma 8 we know

$$\left.\frac{\partial^2 H_{f,t}}{\partial \left(p_{i,t}^f\right)^2}\right|_{\frac{\partial H_{f,t}}{\partial p_{i,t}^f} = 0} < 0$$

Therefore the diagonal terms

$$-\frac{\partial^2 H_{f,t}}{\partial \left(p_{i,t}^f\right)^2}\Bigg|_{\frac{\partial H_{f,t}}{\partial p_{i,t}^f}=0} > 0$$

Now, we need to show that

$$\frac{\partial^2 H_{f,t}}{\partial \left(p_{i,t}^f\right)^2} + \sum_{i,j\in\mathcal{S}} \sum_{f,g\in\mathcal{F}} \frac{\partial^2 H_{f,t}}{\partial p_{i,t}^f \partial p_{j,t}^g} < 0 \text{ for all } i \in \mathcal{S}, f \in \mathcal{F}$$

i.e., to show

$$\frac{\partial^2 H_{f,t}}{\partial \left(p_{i,t}^f\right)^2} + \sum_{f,g} \frac{\partial^2 H_{f,t}}{\partial p_{i,t}^f \partial p_{i,t}^g} < 0 \text{ for all } i \in \mathcal{S}, f \in \mathcal{F}$$

Now,

$$\begin{aligned}
\frac{\partial^2 H_{f,t}}{\partial \left(p_{i,t}^f\right)^2} &= \frac{1}{p_{i,t}^f}\frac{\partial H_{f,t}}{\partial p_{i,t}^f} \cdot \left[\left(-e_{i,t}^f + 1\right) - \frac{A_i^T \cdot \lambda^f}{p_i^f} \cdot \frac{F^{-1}\left(\mu_{i,t}^f\right)}{\int_0^{F^{-1}\left(\mu_{i,t}^f\right)} \tau f(\tau)\, d\tau}\right] \\
&+ \frac{H_{f,t}^i}{p_{i,t}^f} \cdot \left[-\frac{\partial e_{i,t}^f}{\partial p_{i,t}^f} + \frac{A_i^T \cdot \lambda^f}{\left(p_{i,t}^f\right)^2} \cdot \frac{F^{-1}\left(\mu_{i,t}^f\right)}{\int_0^{F^{-1}\left(\mu_{i,t}^f\right)} \tau f(\tau)\, d\tau}\right] \\
&+ \frac{H_{f,t}^i}{p_{i,t}^f} \cdot \left[\frac{\left(A_i^T \cdot \lambda^f\right)^2}{\left(p_i^f\right)^3} \cdot \frac{\int_0^{F^{-1}\left(\mu_{i,t}^f\right)} \tau f(\tau)\, d\tau \frac{1}{f\left(F^{-1}\left(\mu_{i,t}^f\right)\right)} + \left(F^{-1}\left(\mu_{i,t}^f\right)\right)^2}{\left(\int_0^{F^{-1}\left(\mu_{i,t}^f\right)} \tau f(\tau)\, d\tau\right)^2}\right] \qquad (27) \\
&= \frac{H_{f,t}^i}{p_{i,t}^f} \cdot \left[-\frac{\partial e_{i,t}^f}{\partial p_{i,t}^f} + \frac{A_i^T \cdot \lambda^f}{\left(p_{i,t}^f\right)^2} \cdot \frac{F^{-1}\left(\mu_{i,t}^f\right)}{\int_0^{F^{-1}\left(\mu_{i,t}^f\right)} \tau f(\tau)\, d\tau}\right] \\
&+ \frac{H_{f,t}^i}{p_{i,t}^f} \cdot \left[\frac{\left(A_i^T \cdot \lambda^f\right)^2}{\left(p_i^f\right)^3} \cdot \frac{\int_0^{F^{-1}\left(\mu_{i,t}^f\right)} \tau f(\tau)\, d\tau \frac{1}{f\left(F^{-1}\left(\mu_{i,t}^f\right)\right)} + \left(F^{-1}\left(\mu_{i,t}^f\right)\right)^2}{\left(\int_0^{F^{-1}\left(\mu_{i,t}^f\right)} \tau f(\tau)\, d\tau\right)^2}\right]
\end{aligned}$$

The first term in (27) vanishes because

$$\frac{\partial H_{f,t}}{\partial p_{i,t}^f}\Big|_{p_{i,t}^{f,\kappa*}} = 0$$

So,

$$\frac{\partial^2 H_{f,t}}{\partial \left(p_{i,t}^f\right)^2} + \sum_{f,g} \frac{\partial^2 H_{f,t}}{\partial p_{i,t}^f \partial p_{i,t}^g} = -\frac{H_{f,t}^i}{p_{i,t}^f}\left[\frac{\partial e_{i,t}^f}{\partial p_{i,t}^f} + \sum_{g\neq f}\frac{\partial e_{i,t}^f}{\partial p_{i,t}^g}\right] + \frac{H_{f,t}^i}{p_{i,t}^f}\cdot\left[\frac{\frac{A_i^T\cdot\lambda^f}{\left(p_{i,t}^f\right)^2}\cdot F^{-1}\left(\mu_{i,t}^f\right)}{\int_0^{F^{-1}\left(\mu_{i,t}^f\right)}\tau f(\tau)d\tau}\right]$$

$$+\frac{H_{f,t}^i}{p_{i,t}^f}\cdot\left[\frac{\frac{\left(A_i^T\cdot\lambda^f\right)^2}{\left(p_i^f\right)^3}\cdot \int_0^{F^{-1}\left(\mu_{i,t}^f\right)}\tau f(\tau)d\tau\frac{1}{f\left(F^{-1}\left(\mu_{i,t}^f\right)\right)}+\left(F^{-1}\left(\mu_{i,t}^f\right)\right)^2}{\left(\int_0^{F^{-1}\left(\mu_{i,t}^f\right)}\tau f(\tau)d\tau\right)^2}\right]$$

The terms in the first bracket on the right hand side are non-positive from item 8 of assumption A2. The remaining group of terms has been shown to be negative in the proof of Lemma 6 (in particular please refer to (11)) after a change of variable $F(y) = 1 + \left(1+\frac{\mathcal{A}_i^T\cdot\lambda^f}{p_{i,l}^{f,k}}\right)$. Hence

$$-\frac{\partial^2 H_{f,t}}{\partial \left(p_{i,t}^f\right)^2} - \sum_{f,g} \frac{\partial^2 H_{f,t}}{\partial p_{i,t}^f \partial p_{i,t}^g} > 0$$

Therefore, (25) has only one solution. From here we conclude that the VI : find $p^{*\kappa} \in \tilde{\mathcal{K}}$ such that

$$\sum_{f\in\mathcal{F}}\left[\nabla_{p^f} H_f\left(p^{f,\kappa*};\lambda^\kappa;p^{-f,\kappa*}\right)\right]^T\cdot\left(p^f - p^{f,\kappa*}\right) \leq 0 \tag{28}$$

$$\text{for all } p\in\tilde{\mathcal{K}}$$

where

$$\tilde{\mathcal{K}} = \prod_{f\in\mathcal{F}}\left\{p^f : p_{\min}^f \leq p^f \leq p_{\max}^f\right\}$$

(which is same as eqn (43) of the paper) also has one solution which can be expressed as

$$p_{i,t}^{f,\kappa*} = \max\left(p_{i,\min}^f, \min\left(p_{i,t}^{f,\kappa}, p_{i,\max}^f\right)\right)$$

where $p_{i,t}^{f,\kappa}$ is an element of the unique vector that solves (25). Hence the proof. ■

References

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