On-line Appendix

"Managing Clearance Sales in the Presence of Strategic Customers"
by Dan Zhang and William L. Cooper

Proof of Lemma 1. We first prove part 1. If $\beta = 0$, then $h(\theta, \beta) = 0$ for all θ. Hence 0 is the unique fixed point. In the remainder of the proof, we consider the case $\beta > 0$. To establish the existence of a fixed point, observe that $h(\cdot, \beta)$ is continuous and $0 \leq h(\cdot, \beta) \leq 1$, so $h(\cdot, \beta)$ must have at least one fixed point θ on $[0, 1]$.

We have assumed that $c > D(p_1) = 1 - p_1$, so $[c - d_1(\theta)]^+ = c - d_1(\theta)$, and

$$h(\theta, \beta) = \min\left\{\frac{\beta[c - d_1(\theta)]}{D(p_2) - d_1(\theta)}, 1\right\} \tag{58}$$

If $c \geq D(p_2)$ then differentiating (58) with respect to θ shows that $h(\cdot, \beta)$ is non-increasing in θ, and thus the fixed point is unique.

Next consider the case $D(p_1) < c < D(p_2)$. Here (58) reduces to $h(\theta, \beta) = \beta[c - d_1(\theta)]/[D(p_2) - d_1(\theta)]$. Let $f(\theta) = h(\theta, \beta)$. Substituting for $d_1(\theta)$ yields

$$f(\theta) \quad = \quad \beta\left[1 - \frac{D(p_2) - c}{D(p_2) - \alpha D(p_1) - \bar{\alpha} D(r(\theta))}\right] - \theta \tag{59}$$

Showing that the fixed point of $h(\cdot, \beta)$ is unique is equivalent to showing that $f(\theta)$ takes the value 0 exactly once. It suffices by the Poincaré-Hopf Index Theorem (see, e.g., page 48 of Vives 1999) to show that $f(\theta)$ always approaches 0 from above (more formally, $f'(\theta) < 0$ whenever $f(\theta) = 0$). To this end, note first that $f(\theta) < 0$ for $\beta < \theta \leq 1$, therefore $f(\theta)$ can equal 0 only on $[0, \beta]$. We have

$$\frac{df(\theta)}{d\theta} = \begin{cases} \frac{\bar{\alpha}\beta[D(p_2) - c](p_1 - p_2)}{[D(p_2) - \alpha D(p_1) - \bar{\alpha} D(r(\theta))]^2 (1-\theta)^2} - 1 & \text{if } 0 \leq \theta < \widehat{\theta} \\ -1 & \text{if } \widehat{\theta} \leq \theta \leq 1. \end{cases}$$

Using (59) and $D(p) = 1 - p$ for $p \in [0, 1]$, we get

$$g(\theta) = \left.\frac{df(\theta)}{d\theta}\right|_{f(\theta)=0} = \begin{cases} \frac{\bar{\alpha}(\beta-\theta)^2(p_1-p_2)}{\beta(1-\theta)^2(1-p_2-c)} - 1 & \text{if } 0 \leq \theta < \widehat{\theta} \\ -1 & \text{if } \widehat{\theta} \leq \theta \leq 1. \end{cases}$$

It can be checked that $g(\theta)$ is strictly decreasing on $[0, \beta]$. Furthermore,

$$g(0) = \frac{\bar{\alpha}\beta(p_1 - p_2)}{1 - p_2 - c} - 1.$$

If $g(0) < 0$, then the fact that $g(\theta)$ is decreasing implies that $df(\theta)/d\theta|_{f(\theta)=0} < 0$; i.e., $f(\theta)$ approaches 0 only from above. Hence $f(\theta)$ only take the value 0 once.

If $g(0) \geq 0$, we show that it is only possible for $f(\theta)$ to take the value 0 where $g(\theta) < 0$. Let $\theta' = \min\{\theta : f(\theta) = 0\}$. Because

$$f(0) = \beta \left[1 - \frac{D(p_2) - c}{D(p_2) - D(p_1)}\right] > 0,$$

it follows that $f(\theta)$ must approach 0 from above at θ'. This implies that $g(\theta') \leq 0$. Since $g(\theta)$ is strictly decreasing, we have $g(\theta) < 0$ for $\theta > \theta'$. Hence $f(\theta)$ takes the value 0 only once. This completes the proof of part 1.

For part 2, we consider two cases.

Case 1, $D(p_1) < c < D(p_2)$: To show Θ is an interval, it suffices to show that the implicit function $\theta(\beta)$ determined by (23) is continuous. Lemma 1 implies that the function $\theta(\beta)$ is a well-defined function of β. From (23) and the expression for $h(\theta, \beta)$ in (59), $\theta(\beta)$ has an inverse given by $\beta(\theta) = \theta/[1 - t(\theta)]$, where

$$t(\theta) = \frac{D(p_2) - c}{D(p_2) - \alpha D(p_1) - \bar{\alpha} D(r(\theta))}.$$

Since $r(\theta)$ is continuous in θ and $D(\cdot)$ is continuous, we have that $\beta(\theta)$ is continuous. Hence $\theta(\beta)$ is continuous. The fact that Θ is an closed interval follows from the fact that the set of β values $[0, 1]$ is closed. Furthermore, it is easy to see that $\theta(0) = 0$ hence $0 \in \Theta$. This completes the proof for case 1.

Case 2, $c \geq D(p_2)$: For $c \geq D(p_2)$, we have $t(\theta) \leq 0$ and

$$\begin{aligned} h(\theta, \beta) &= \min\left\{\beta\left[1 - t(\theta)\right], 1\right\} \\ &= \begin{cases} \beta\left[1 - t(\theta)\right] & 0 \leq \beta \leq 1/[1 - t(\theta)] \\ 1 & \beta > 1/[1 - t(\theta)]\,. \end{cases} \end{aligned}$$

From the definition of Θ,

$$\Theta \supseteq \big\{\theta : h(\theta, \beta) = \theta \text{ for some } \beta \in \big[0, 1/[1 - t(\theta)]\big]\big\} \equiv \Theta_1.$$

Arguments similar to those used in the proof of case 1 can be used to show that Θ_1 is an interval of the form $[0, \bar{\theta}]$ for some $\bar{\theta} \in [0, 1]$. Furthermore, it can be checked that 0 and 1 belong to Θ_1. Therefore, $[0, 1] \subseteq \Theta_1 \subseteq \Theta$. Using the fact that $\Theta \subseteq [0, 1]$, we conclude that $\Theta = [0, 1]$. This completes the proof for case 2. ■

Proof of Lemma 2. Lemma 1 shows that if $c \geq 1 - p_2$, then $\bar{\theta} = 1$. In the following, we consider the case $1 - p_1 < c < 1 - p_2$. Starting from the definition of $h(\theta, \beta)$ in (23) and after some algebra, we obtain

$$h(\theta, \beta) = \begin{cases} h_1(\theta, \beta) & \text{if } 0 \leq \theta \leq \widehat{\theta} \\ h_2(\theta, \beta) & \text{if } \widehat{\theta} < \theta \leq 1 \end{cases}$$

where

$$\begin{aligned} h_1(\theta,\beta) &= \beta\left[1-\frac{(1-p_2-c)(1-\theta)}{(p_1-p_2)(1-\theta\alpha)}\right] \\ h_2(\theta,\beta) &= \beta\left[1-\frac{1-p_2-c}{1-\alpha+\alpha p_1-p_2}\right]. \end{aligned}$$

Since $h(\theta,\beta)$ is increasing in β, and $h(\cdot,\beta)$ has a unique fixed point for each β, a geometrical argument can be used to show the fixed point of $h(\cdot,\beta_1)$ is greater than the fixed point of $h(\cdot,\beta_2)$ if $\beta_1>\beta_2$. Therefore, $\bar{\theta}$ is the solution of the equation $h(\theta,1)=\theta$.

To solve $h(\theta,1)=\theta$, we first consider $h_1(\theta,1)=\theta$. If $\alpha=0$, we obtain $\theta=1>\widehat{\theta}$. So there is no solution to $h(\theta,1)=\theta$ on $[0,\widehat{\theta}]$. If $0<\alpha\leq 1$, we obtain

$$\theta=\frac{c-1+p_1}{\alpha(p_1-p_2)}. \tag{60}$$

If $c\leq c_1$, then the right hand side of (60) is no larger than $\widehat{\theta}$ in which case (60) gives the solution of $h(\theta,1)=\theta$. If $c>c_1$, the right hand side of (60) is greater than $\widehat{\theta}$, and therefore is not a solution of $h(\theta,1)=\theta$ on $[0,\widehat{\theta}]$.

Next, we consider $h_2(\theta,1)=\theta$. If $0\leq\alpha\leq 1$, we obtain

$$\theta=\frac{c+\alpha p_1-\alpha}{\alpha p_1-p_2+1-\alpha}.$$

Note that when $c>c_1$, the right hand side above is greater than $\widehat{\theta}$. ■

Proof of Proposition 8. The proof is largely based on the proof in the infinite capacity case. In the following, we provide a brief outline. By (36)

$$\begin{aligned} w^{\text{FP}} &= \max_{0\leq\theta\leq\bar{\theta}}\Psi(\theta) \\ &= \begin{cases} \max_{0\leq\theta\leq\bar{\theta}}\Psi(\theta) & \text{if } \bar{\theta}\leq\widehat{\theta} \\ \max\left\{\max_{0\leq\theta\leq\widehat{\theta}}\Psi(\theta),\max_{\widehat{\theta}\leq\theta\leq\bar{\theta}}\Psi(\theta)\right\} & \text{if } \bar{\theta}>\widehat{\theta}. \end{cases} \end{aligned}$$

From the proof of Proposition 4, $\Psi(\theta)$ is linear increasing for $\theta\in[\widehat{\theta},1]$; hence

$$w^{\text{FP}}=\begin{cases} \max_{0\leq\theta\leq\bar{\theta}}\Psi(\theta) & \text{if } \bar{\theta}\leq\widehat{\theta} \\ \max\left\{\max_{0\leq\theta\leq\widehat{\theta}}\Psi(\theta),\Psi(\bar{\theta})\right\} & \text{if } \bar{\theta}>\widehat{\theta}. \end{cases}$$

Let $\bar{x}=(p_1-\bar{\theta}p_2)/(1-\bar{\theta})$. Observe that $\bar{x}\leq 1$ for $\theta\in[0,\widehat{\theta}]$ and $\bar{x}>1$ for $\theta\in[\widehat{\theta},1]$. As in the proof of Proposition 4, the optimization problem (34) can be written as

$$w^{\text{FP}}=\begin{cases} R+(p_1-p_2)\max_{p_1\leq x\leq\bar{x}}F(x) & \text{if } \bar{x}\leq 1 \\ R+(p_1-p_2)\max\left\{\max_{p_1\leq x\leq 1}F(x),(1-\bar{\theta})\Omega(\bar{\theta})\right\} & \text{if } \bar{x}>1, \end{cases} \tag{61}$$

where $F(\cdot)$ is defined in (43). There it is shown that $F(x)$ is concave on $[p_1, 1]$.
Part (i), $\alpha = 1$: Here all customers are myopic and it is straightforward to show that $\theta^{**} = \bar{\theta}$.
Part (ii), $\alpha = 0$: Here all customers are strategic. From Lemma 2, $\bar{\theta} = c/(1 - p_2) > \widehat{\theta}$. After some algebra, we obtain

$$w^{\mathrm{FP}} = R + (p_1 - p_2) \max\left\{1 - p_1 - p_2, -\frac{(1 - p_2)p_2}{\bar{x} - p_2}\right\}.$$

The result in (38) can now be easily verified.
Part (iii), $0 < \alpha < 1$: An optimal solution for (61) can be obtained based on the solution in the infinite capacity case. Once we have x^{**} that maximizes (61), we can get an optimal solution to (34) by taking $\theta^{**} = (x^{**} - p_1)/(x^{**} - p_2)$. The details are omitted. ■

References

Vives, X. 1999. *Oligopoly Pricing: Old Ideas and New Tools*. MIT Press, Cambridge, MA.